Zentrum für
Technomathematik

Fachbereich 3
Mathematik und Informatik

Optimally Matched Wavelets

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Overview

1. Problem and applications
2. Matching Wavelets
3. Outlook

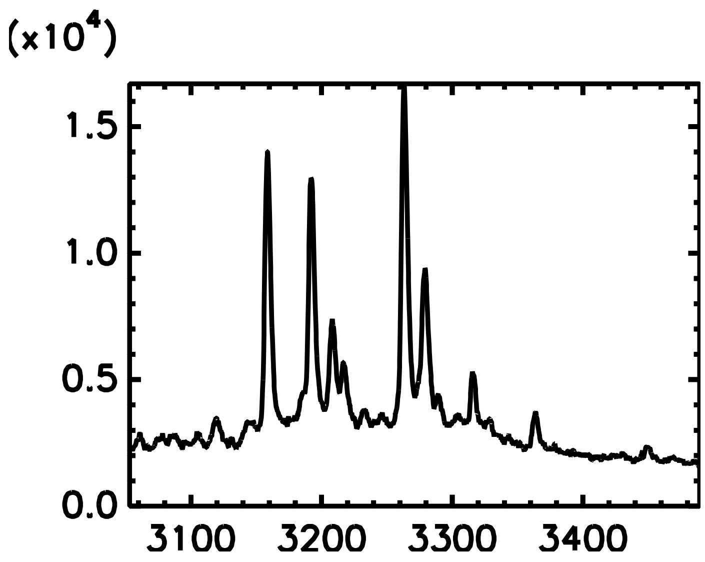
Problem

Signal contains the same pattern at different scales.

- Pattern matching - retrieve patterns
- Denoising, Compression - remove any non-matching structure

Applications

- Detection of component (ball-bearing) wearout by observing the current of an engine
- Detection of pollutions in rotor spinning machines
- Decomposition of (audio) signals into time-frequency atoms
- Extraction of peaks from a mass spectrogram



Obvious Method

Multi-scale correlation alias *Continuous Wavelet Transform*

Advantages:

- Multiple scales
- Translation invariant
- Fine frequency sampling
- Mathematically reversible
- Weak restrictions on pattern

Disadvantages:

- Not numerically reversible
- Slow

Alternative Method

Discrete wavelet transform

Advantages:

- Multiple scales
- Reversible
- Fast

Disadvantages:

- Strong restrictions on pattern
- Translation dependent
- Coarse frequency sampling

Discrete Wavelet Transform

The *Discrete Wavelet Transform* coincides with *Subband Coding*. It is described by four filters $h, g, \widetilde{h}, \widetilde{g}$.

- Analysis: h, g
- Synthesis: $\widetilde{h}, \widetilde{g}$

Dependencies for perfect reconstruction

- Choice of h is limitted.
- Choice of h limits choice of g.
- Choice of h and g determines $\widetilde{h}$ and $\widetilde{g}$ and thus fixes the transform.

Continuous view on DWT

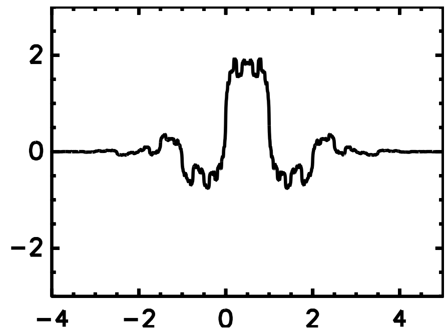
- Discrete Wavelet Transform = Filtering
- Alternative interpretation: Expansion into a wavelet base

- Filters $h, \widetilde{h}$ correspond to the *generator functions* $\varphi, \widetilde{\varphi}$, which are refinable.
- Filters $g, \widetilde{g}$ correspond to the *wavelet functions* $\psi, \widetilde{\psi}$, which are finite linear combinations of translates of $\varphi, \widetilde{\varphi}$, respectively.

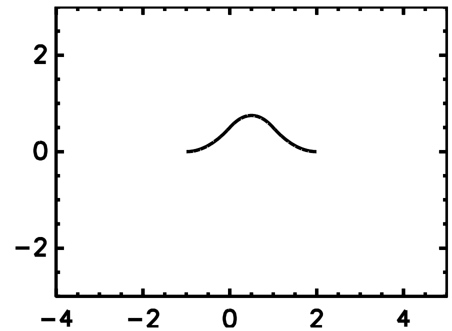
Wavelet basis functions

Basis of Cohen, Daubechies, Feauveau of order 3,5

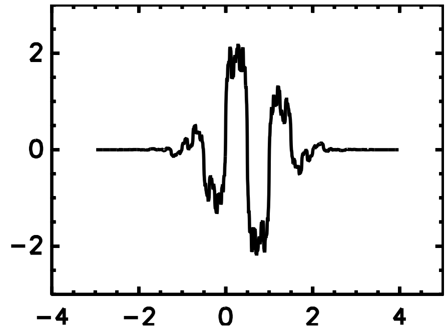
Generators: $h \sim \varphi$



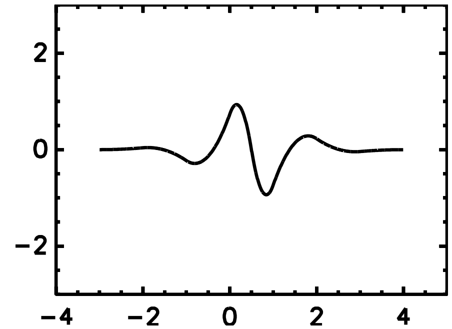
$\widetilde{h} \sim \widetilde{\varphi}$



Wavelets: $g \sim \psi$



$\widetilde{g} \sim \widetilde{\psi}$



Pattern meets Wavelet

Is it possible to create a discrete wavelet similar to a given pattern?

Lifting

Lifting step: Transform a perfect reconstruction filter bank into another such filter bank by a filter s.

reconstructable filter bank		reconstructable filter bank
h, g	$\longrightarrow$	h, g_s

We use this principle for the design of wavelet filters.

Matching wavelets using Lifting

1. Choose a generator φ with mask h
   $\Rightarrow$ Determines smoothness of the wavelet, too.
2. Choose a wavelet ψ with mask g.
   Filters g and h must allow for perfect reconstruction.
3. Find a lifting step that leads to g_s whose wavelet $\psi_{c,s}$ optimally matches pattern f.
4. $\Rightarrow$ Reduction to a simple least squares problem!
5. Structure of refinable functions allows for quick computation of normal equations.

Optimization target:

$$\underset{c,s}{\operatorname{argmin}} \|\psi_{c,s} - f\|_2 \qquad \text{with} \qquad \psi_{c,s} = c \cdot \psi + s * \varphi$$

The set $\{\psi_{c,s} : c \in \mathbb{R}, s \in \mathbb{R}^{\mathbb{Z}}\}$ forms a linear space.

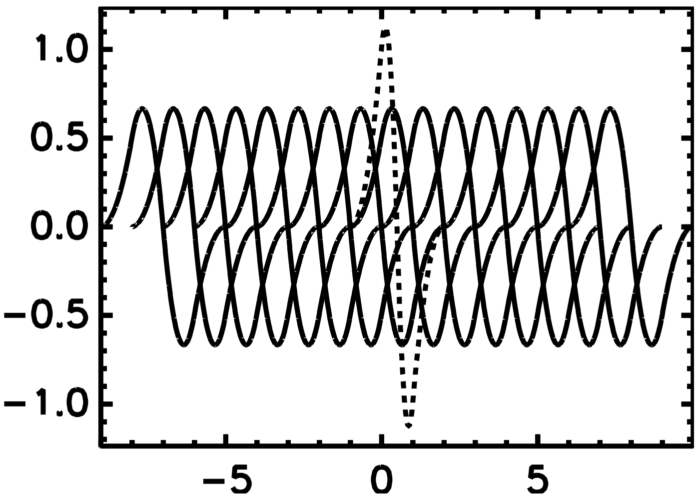
Example base

Most simple choice: The Binomial mask

$$h = (1, 1)^n$$

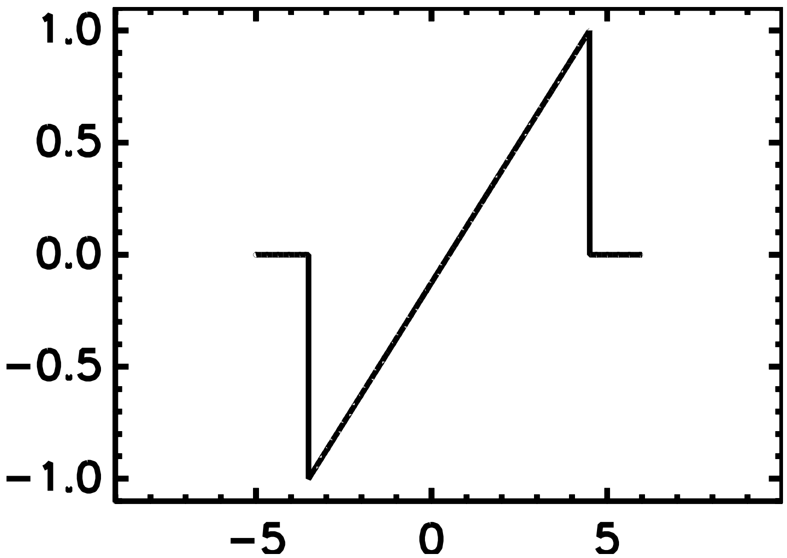
corresponds to B-Spline of order $n - 1$.

Example $n = 3$
Basis functions:

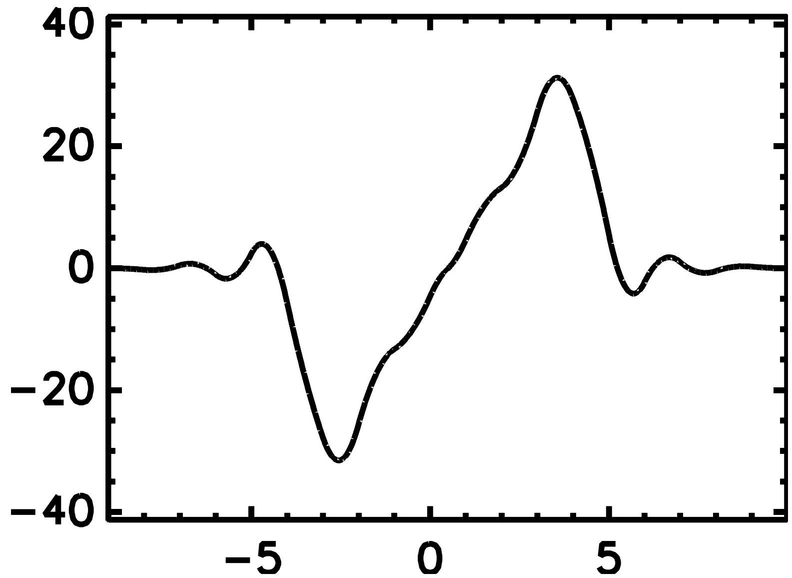


Nice approximation

Original

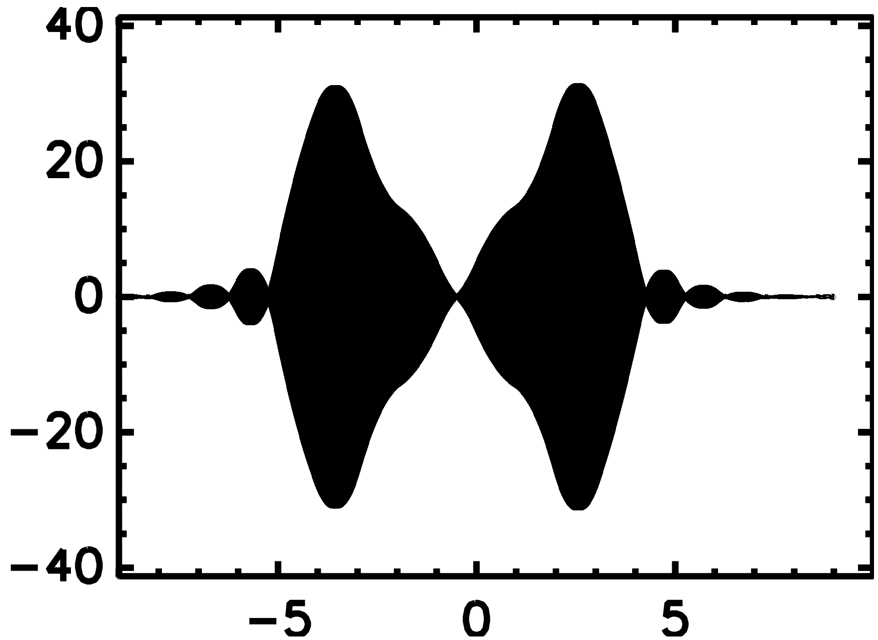


Approximation



Ugly dual generators

A nicely matched wavelet leads to an ugly dual generator.



Low smoothness means bad numerical properties.

Obtaining smoothness for dual functions

There are also approaches to solve this problem but two solutions are too much for one talk!

Remaining problems

- Coarse pattern of DWT coefficients.
  The recursive dyadic grid of the DWT coefficients may not match the positions where the patterns actually occur. Omitting subsampling and using interim scales solves this problem but causes big output. Is it possible to solve that by nearly translation invariant filters?
- Analysis and synthesis wavelet differ.
  May it be better to use orthogonal wavelets? Orthogonal wavelets are hard to control, they do not allow for a simple least square approach. Orthogonal wavelets can not be symmetric. Can one benefit from the additional degrees of freedom the translation invariant DWT gives to filter design?